Session 8

Exercise 1: Classification of electromagnetic problems

Given the general form $a\Phi_{xx} + b\Phi_{xy} + c\Phi_{yy} + d\Phi_x + e\Phi_y + f\Phi = g$,

a) when is a PDE called linear and when nonlinear?

b) when is a PDE called homogeneous and when inhomogeneous?

c) Align the general form with the operator notation $L\Phi = g$ and give the expression for L.

Exercise 2: Classification of electromagnetic problems

Classify the following partial differential equations as elliptic, parabolic or hyperbolic.

a) $\Phi_{xx} + 2\Phi_{xy} + 5\Phi_{yy} = 0$

b) $(y^2 + 1)\Phi_{xx} + (x^2 + 1)\Phi_{yy} = 0,\ x, y \in \mathbf{R}$

c) $x^2\Phi_{xx} - 2xy\Phi_{xy} + y^2\Phi_{yy} + x\Phi_x + y\Phi_y = 0$

Exercise 3: Classification of electromagnetic problems

For the following problems, give the partial differential equations and perform the classification

a) Wave propagation in one dimension

b) Temperature distribution in material in one dimension

c) Electric potential from a charge distribution

State if the solution region is usually open or closed.

Exercise 4: Finite Difference Methods

From a Taylor's series approach for $f(x + \triangle x)$ and $f(x - \triangle x)$ show that

a) $f'(x) \approx \frac{f(x_0+\triangle x)-f(x_0-\triangle x)}{2(\triangle x)}$, where $f'(x) \in \mathcal{O}(\triangle x)^3$

a) $f''(x) \approx \frac{f(x_0+\triangle x)-2f(x_0)+f(x_0-\triangle x)}{(\triangle x)^2}$, where $f''(x) \in \mathcal{O}(\triangle x)^4$

Hints: Write the Taylor's series expressions for $f(x + \triangle x)$ and $f(x - \triangle x)$ and perform a superposition to obtain the first and second order derivative. Drop terms and observe their order to get the desired expressions.

Exercise 5: Finite Difference Approximation of the one-dimensional diffusion equation

Derive the finite difference approximation of the one-dimensional diffusion partial differential equation

$$k\frac{\partial\Phi}{\partial t} = \frac{\partial^2\Phi}{\partial^2 x}$$

where k $[\frac{W}{m\cdot K}]$ is the thermal conductivity and a constant. Let $\Phi(x, y)$ become $\Phi(i, j)$, where $x = i \cdot \triangle x$ and $i = 0, 1, 2, ..., n$, $t = j \cdot \triangle t$ and $j = 0, 1, 2, ..., t_{max}$.

Hints: Use the forward difference formula in this exercise.

Exercise 6: Matlab Exercise: Finite Difference Approximation of the one-dimensional diffusion equation (explicit form)

For $r = \frac{\triangle t}{k(\triangle x)^2}$, the finite difference approximation of the one-dimensional diffusion equation can be written in explicit form

$$\Phi(i, j+1) = r\Phi(i+1, j) + (1-2r)\Phi(i, j) + r\Phi(i-1, j)$$

a) Write a Matlab program to simulate a one-dimensional temperature distribution.

b) Perform a simulation of a 10 m long steel $k = 50$ $[\frac{W}{m\cdot K}]$ and aluminum $k = 250$ $[\frac{W}{m\cdot K}]$ rod. The diameter shall be negligible due to the length of the rod. The rod is constantly heated at one end at a temperature of $1200°C$ in an ambient temperature of $20°C$. Initially, the rod temperature equals the ambient temperature.

   Hints: Let $0 < r \leq 1/2$ to obtain stable results.

c) Show the development of the temperature along the rod over time.

   Hints: Read the function getframe() in the Matlab documentation.

d) Track the temperatures at the middle and end of the rod over time and show the graphs.

e) At merely $800°C$, we assume steel to lose half its strength.
   When is this temperature reached in the middle of the steel rod?
   When is this temperature reached at the middle of the aluminum rod?

Exercise 7: Matlab Exercise: Finite Difference Approximation of the one-dimensional diffusion equation (implicit form)

An implicit formulation of the finite difference approximation of the diffusion equation was proposed by Crank and Nicholson in 1974. There, the second order derivative $\partial^2\Phi/\partial x^2$ is replaced by the average of the central difference formulas on the j-th and $j+1$-th time rows.

a) Perform the replacement and show that

$$k\frac{\Phi(i,j+1)-\Phi(i,j)}{\triangle t} = \frac{1}{2}\left[\frac{\Phi(i+1,j)-\Phi(i,j)+\Phi(i-1,j)}{(\triangle x)^2}+ \frac{\Phi(i+1,j+1)-2\Phi(i,j+1)+\Phi(i-1,j+1)}{(\triangle x)^2}\right]$$

b) Show that the expression can be rewritten as

$$-r\Phi(i-1,j+1)+2(1+r)\Phi(i,j+1)- r\Phi(i+1,j+1) = r\Phi(i-1,j)+ 2(1-r)\Phi(i,j)+r\Phi(i+1,j)$$

Hints: See that all addends on the left depend on $j+1$ while all on the right depend on j. All terms on the right are known at time $j+1$.

c) Draw the computational molecule for all finite $r \neq 1$ and $r = 1$.

Hints: Refer to the script module 8, pages 41 to 43.

d) Set up the linear equation system for $r = 1$ and $i = 1, .., 4$ by considering the boundary conditions and initial conditions from the previous exercise. Write a Matlab program to solve the equation system and simulate a one-dimensional temperature distribution.

Hints: For $i = 1, .., 4$ we get 4 equations from the expression in b) considering the boundary conditions $\Phi(1-1,*) = 1500°C$, $\Phi(4+1,*) = 20°C$ and initial condition $\Phi(i,0) = 20°C$.

e) Perform the same simulations as in the previous exercise.